Averaging for minimum time control problems and applications

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Coordinates (I, φ)

- $I \in U(\subset \mathbb{R}^5)$: geometric elements of the ellipse,
- $\varphi \in S^1$: position on the orbit.

Equations of motion In first approximation,

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I) > 0$$

I is a slow variable compared to φ.

$$\min_{t_f, |u| \leq 1} \quad t_f$$

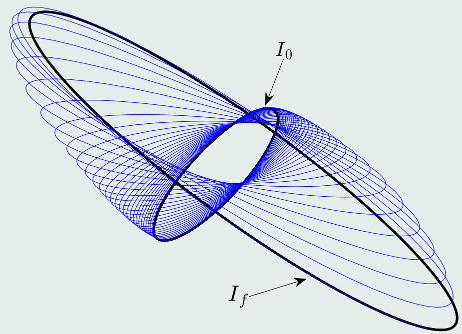
$$\text{s.t.} \quad \dot{I} = \varepsilon F_0(I, \varphi, \varepsilon) + \varepsilon \sum_{i=1}^{3} u_i F_i(I, \varphi, \varepsilon), \qquad I(0) = I_0$$

$$\dot{\varphi} = \omega(I) + \varepsilon G_0(I, \varphi, \varepsilon) + \varepsilon \sum_{i=1}^{3} u_i G_i(I, \varphi, \varepsilon), \quad I(t_f) = I_f.$$

$0 < \varepsilon \ll 1$, and I_0, I_f are given. $\varphi(0)$ and $\varphi(t_f)$ are free.
$F_i(\cdot, \varphi, \cdot), G_i(\cdot, \varphi, \cdot)$ $i = 0 \ldots 3$ are **2π-periodic** w.r.t. φ, $u \in \mathbb{R}^3$.

F_0, G_0 modeled perturbations

- high order terms in the gravitational potential of the Earth,
- luni-solar perturbation, ...



Let $f(X,\psi,\varepsilon), g(X,\psi,\varepsilon)$ be two vector fields **2π-periodic** w.r.t. $\psi \in S^1$.

Unperturbed system $\dot{X} = 0, \quad \dot{\psi} = \omega(X)$

Perturbed system

$$\dot{X} = \varepsilon f(X,\psi,\varepsilon), \quad \dot{\psi} = \omega(X) + \varepsilon g(X,\psi,\varepsilon)$$

where $X \in C \subset \mathbb{R}^n$.There are $C' \subset C$ and $\tau > 0$ s.t.

$$Y' = \overline{f}(Y), \quad \overline{f}(X) = \frac{1}{2\pi}\int_0^{2\pi} f(X,\psi,\varepsilon = 0)\,\mathrm{d}\psi,$$

have a well-defined solution defined on $[0,\tau]$ for every initial condition $Y(0) \in C'$.

Theorem 1. $\exists \varepsilon_0 > 0$, $0 < \varepsilon < \varepsilon_0$ *and for every initial condition in* $C' \times S^1$, *the perturbed system has a solution on* $[0,\tau/\varepsilon]$ *s.t.*

$$|X(t) - Y(\varepsilon t)| = O(\varepsilon)$$

PMP: a minimizing trajectory is projection of an extremal $(I(.), \varphi(.), p_I(.), p_\varphi(.))$ solution of the Hamiltonian system

$$H(I, \varphi, p_I, p_\varphi, \varepsilon) = p_\varphi \omega(I) + \varepsilon K_0(I, \varphi, p_I, p_\varphi, \varepsilon),$$

$$K_0 := H_0 + \sqrt{\sum_{i=1}^{3} H_i^2}, \;\; H_i(I, \varphi, p_I, p_\varphi, \varepsilon) = p_I F_i(I, \varphi, \varepsilon) + p_\varphi G_i(I, \varphi, \varepsilon), \;\; i = 0, ..., 3.$$

Extremal system

$$\dot{I} = \varepsilon \frac{\partial K_0}{\partial p_I}, \quad \dot{p_I} = -p_\varphi \omega' - \varepsilon \frac{\partial K_0}{\partial I}$$

$$\dot{\varphi} = \omega + \varepsilon \frac{\partial K_0}{\partial p_\varphi}, \quad \dot{p_\varphi} = -\varepsilon \frac{\partial K_0}{\partial \varphi}$$

Theorem 2. *Normalizing on $\boldsymbol{H = \varepsilon}$ (normal extremals), we have*

$$p_\varphi = -\varepsilon h(I, \varphi, p_I, \varepsilon)$$

$$0 = \frac{\partial}{\partial p_I}\left(H(I,\varphi,p_I,-\varepsilon h(I,\varphi,p_I,\varepsilon),\varepsilon) - \varepsilon\right) = \varepsilon\frac{\partial K_0}{\partial p_I} - \varepsilon\frac{\partial h}{\partial p_I}\frac{\partial H}{\partial p_\varphi}$$

$\Rightarrow\ \dot{I} = \varepsilon\frac{\partial h}{\partial p_I}\,\dot{\varphi}$, and similarly, we have: $\dot{p}_I = -\varepsilon\frac{\partial h}{\partial I}\,\dot{\varphi}$.

$\Rightarrow$ Reparametrization w.r.t.: $s = \varepsilon(\varphi - \varphi_0)$.

Definition 3. *The* **average** *boundary value problem is*

$$\frac{\mathrm{d}\overline{I}}{\mathrm{d}s} = \frac{\partial\overline{h}}{\partial p_I}, \qquad \frac{\mathrm{d}\overline{p}_I}{\mathrm{d}s} = -\frac{\partial\overline{h}}{\partial I}$$
$$\overline{I}(0) = I_0, \quad \overline{I}(\overline{s}_f) = I_f, \quad \overline{h} = 0$$

where

$$\overline{h}(I,p_I) := \frac{1}{2\pi}\int_0^{2\pi} h(I,\varphi,p_I,\varepsilon{=}0)\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi}\frac{K_0-1}{\omega}(I,\varphi,p_I,p_\varphi{=}0,\varepsilon{=}0)\mathrm{d}\varphi.$$

Let

$$\overline{S}(s_f, p_0) = (\overline{I}(s_f, p_0) - I_f, \overline{h}(I_0, p_0))$$

be the shooting function associated with the average dynamic

$$\frac{\mathrm{d}\overline{z}}{\mathrm{d}s} = \overrightarrow{\overline{h}}(\overline{z}(s)), \quad \overline{z} = (\overline{I}, \overline{p}_I)$$

Remark 4. *$\overline{h}$ is smooth outside $\overline{\Sigma}$ where*

$$\Sigma = \{(I, p_I, \varphi) \in T^*M \times S^1 \mid p_I \cdot F_i(I, \varphi, p_I, \varepsilon) = 0,\ i = 1, 2, 3\}$$

in

$$\overline{\Sigma} = \Pi(\Sigma), \quad \Pi : T^*M \times S^1 \to T^*M.$$

For $\varepsilon > 0$, consider

$$S_\varepsilon(s_f, p_0) = (I(s_f, p_0) - I_f, \overline{h}(I_0, p_0))$$

be the shooting function associated with the non-average dynamic

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \overrightarrow{h}(s/\varepsilon, z(s), \varepsilon), \quad z = (I, p_I).$$

Remark 5. *Note that solutions of $S_\varepsilon(s_f, p_0) = 0$ doesn't define strict extremals in the sense that the true conditions*

$$h(\varphi_0, z_0, \varepsilon) = h(\varphi_f, z_f, \varepsilon) = 0$$

are replace by

$$\overline{h}(z_0) = 0.$$

They are called "quasi-extremals".

Theorem 6. *Let $(I^\varepsilon(.))_\varepsilon$ be a family of non average trajectories (with "quasi-extremals" lifts) converging as $\varepsilon \to 0$ with fixed extremities I_0, I_f. Then the limit is solution of the average boundary value problem.*

Theorem 7. *Let $\overline{I}(.)$ be a trajectory of the average system which doesn't have conjugate points.*
Then, there exists a family of non average trajectories $(I^\varepsilon(.))_\varepsilon$ converging to $\overline{I}(.)$ as $\varepsilon \to 0$ satisfying the boundary values: $I^\varepsilon(0) = I_0$ and $I^\varepsilon(t_f) = I_f + O(\varepsilon)$ and under further assumptions, there exists (φ_0, p_0) s.t.

$$h(\varphi_0, z(0), \varepsilon) = 0 \quad \textit{and} \quad h(s_f/\varepsilon + \varphi_0, z(s_f), \varepsilon) = 0.$$

Remark 8. *Numerical use: The solution of the average system can be used to* **initialize** *the shooting function of the non average system.*

Lemma 9. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function with a regular zero at $x = 0$, and let $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, $\varepsilon > 0$, be continuous functions converging uniformly towards f on a neighborhood of the origin when $\varepsilon \to 0$.*
There exists $\varepsilon_0 > 0$ and $x : [0, \varepsilon_0] \to \mathbb{R}^n$ continuous at $\varepsilon = 0$, s.t. $x(0) = 0$ and

$$f_\varepsilon(x(\varepsilon)) = 0, \qquad \varepsilon \in (0, \varepsilon_0].$$

Proof of the convergence Theorem 7. Let $(\overline{s}_f, \overline{p}_0)$ the solution of the average system ($\overline{S}(\overline{s}_f, \overline{p}_0) = 0$).

1. Apply Lemma 9 with $f = \overline{S}$ and $f_\varepsilon = S_\varepsilon$. Then there exists $p_0(\varepsilon) \to \overline{p}_0$ s.t. $\overline{h}(I_0, p_0(\varepsilon)) = 0$.

2. Assume

$$\exists\overline{\varphi}_0 \in S^1, \quad h(\overline{\varphi}_0, \overline{z}_0, \varepsilon = 0) = 0, \quad \frac{\partial h}{\partial \varphi}(\overline{\varphi}_0, \overline{z}_0, \varepsilon = 0) \neq 0, \quad (\overline{z}_0 = (I_0, \overline{p}_0)),$$

we construct $\varphi_0(p, \varepsilon)$ and we apply again Lemma 9 with $S_\varepsilon(\varphi_0, s_f, p_0)$ associated to

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \overrightarrow{h}\,(s/\varepsilon + \boldsymbol{\varphi_0}, z(s), \varepsilon)$$

to satisfy the condition $\boldsymbol{h(\varphi_0, z(0), \varepsilon) = 0}$ for the non average extremals.

3. Assume

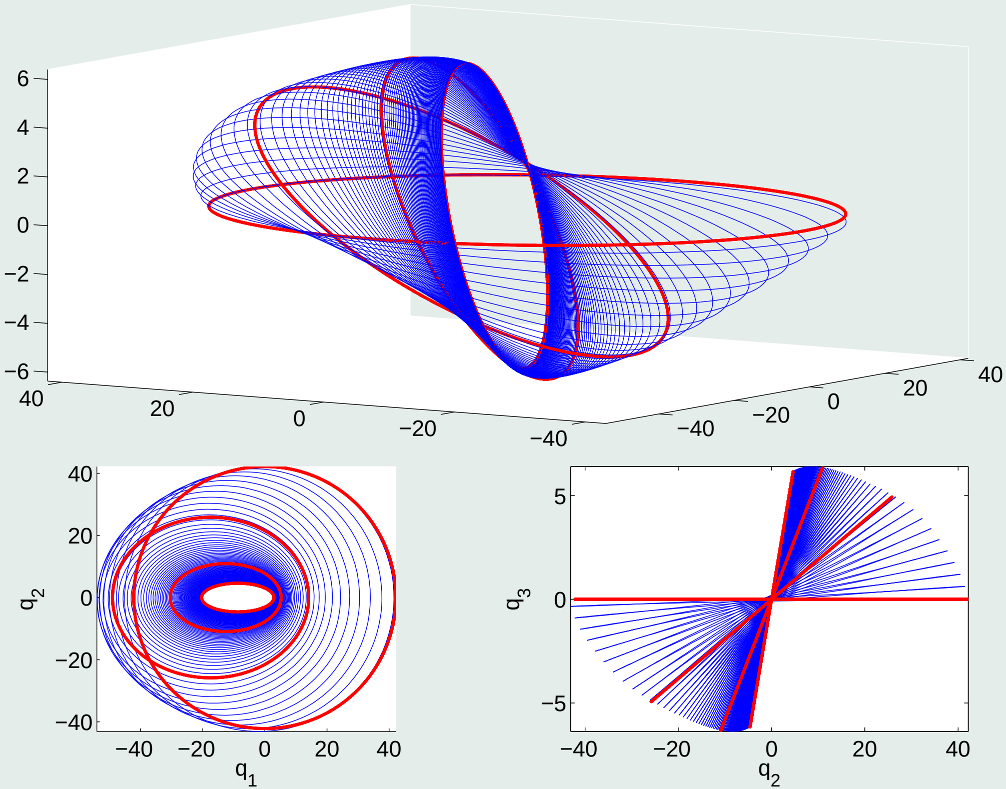
$$\exists\overline{\varphi}_f \in S^1, \quad h(\overline{\varphi}_f, \overline{z}_f, \varepsilon = 0) = 0, \quad \frac{\partial h}{\partial \varphi}(\overline{\varphi}_f, \overline{z}_f, \varepsilon = 0) \neq 0, \quad (\overline{z}_f = \overline{z}(\overline{s}_f, p_0)),$$

we construct $\varphi_f(\varepsilon)$ (Brouwer fix point theorem) and consider

$$\widetilde{s}_f(\varepsilon) = s_f(\varepsilon) + \varepsilon\varphi_f(\varepsilon),$$

so that

$$h(\widetilde{s}_f/\varepsilon + \varphi_0, z(\widetilde{s}_f), \varepsilon) = 0 \text{ and } I(\widetilde{s}_f) = I_f + O(\varepsilon).$$



(red) Some average orbits of the average system.
(blue) Non average trajectory in Cartesian coordinates.

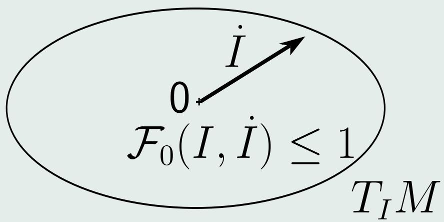
Finsler norm $\mathcal{F} : TM \to \mathbf{R}$ smooth on $TM\backslash 0$ s.t.

- $\mathcal{F}(x, \lambda v) = \lambda \mathcal{F}(x, v)$, $\lambda > 0$
- $\partial^2 \mathcal{F}^2(x, v)/\partial v^2 > 0$

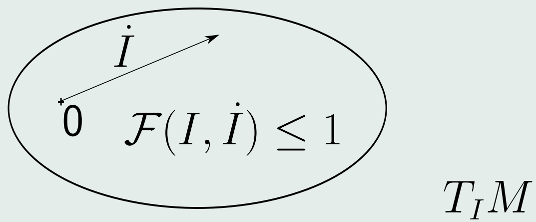
Finsler co-norm $F^* : T^*M \to \mathbf{R}$ smooth on $T^*M\backslash 0$ s.t.

- $F^*(x, \lambda p) = \lambda F^*(x, p),\ \lambda > 0,$
- $\partial^2 (F^*)^2(x, p)/\partial p^2 > 0$

Theorem 10. *If* $\mathrm{rang}\ \{\partial^j F_i(I, \varphi, \varepsilon = 0)/\partial \varphi^j,\ i = 1, \ldots, 3,\ j \geq 0\} = 6,$ *then* $\overline{K}_0$ *defined a symmetric Finsler co-norm.*



$\mathcal{F}_0$ Symmetric Finsler norm



$\mathcal{F}$ Non-symmetric Finsler norm

Earth oblateness perturbation

$$
\begin{aligned}
H &= p_\varphi \omega(I) + \varepsilon_{J_2} H_0(I, p_I, \varphi) + \varepsilon_{th} K_0(I, p_I, \varphi, p_\varphi, \varepsilon) \\
&= p_\varphi \omega(I) + \varepsilon \Big[\underbrace{\lambda H_0(I, p_I, \varphi) + (1-\lambda) K_0(I, p_I, \varphi, p_\varphi, \varepsilon)} \Big] \\
&= p_\varphi \omega(I) + \varepsilon \qquad\qquad K^\lambda(I, p_I, \varphi, p_\varphi, \varepsilon)
\end{aligned}
$$

where $\varepsilon = \varepsilon_{J2} + \varepsilon_{th},\ \lambda = \varepsilon_{J2}/\varepsilon$.

We denote by $\overline{K}^\lambda$ the average of K^λ w.r.t. φ.

• *Q :* Is $(I, p_I) \mapsto \overline{K}^{\lambda}(I, p_I) = \lambda \overline{H}_0(I, p_i) + (1-\lambda)\overline{K}_0(I, p_I)$ a Finsler co-norm ?

Lemma 11. *For $I \in M$ fixed, consider the (unique) one-form $\overline{F_0}^{\star,\lambda}(I)$ solution of*

$$\overline{F_0}^{\star,\lambda}(I) = \underset{p \in \mathbb{R}^n}{\textit{argmin}} \left[1/2\,(1-\lambda)^2 \overline{K}_0(I, p)^2 - \lambda \langle p, \overline{F}_0(I) \rangle \right].$$

Proposition 12. *Let $I \in M$ fixed and λ_c s.t. $\overline{K}_0^{\lambda_c}(I, \overline{F_0}^{\lambda_c \star}(I)) = 1$. For all $\lambda < \lambda_c$,*

$$T_I^{\star} M \ni p \mapsto \overline{K}^{\lambda}(I, p) \in \mathbb{R}^+$$

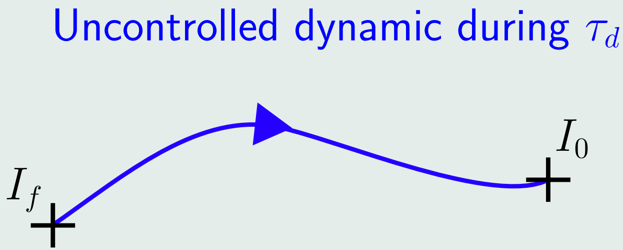
is a Finsler co-norm (non-symmetric).

Proposition 13. *λ_c can be computed as*

$$\lambda_c(I) = \frac{1}{\overline{K}_0(I, p^{\star}) + 1},$$

where $p^{\star} = \langle \pi^{\star}, \overline{F}_0(I) \rangle\, \pi^{\star},\ \pi^{\star} = \underset{\pi | \overline{K}_0(I, \pi) = 1}{\textit{argmax}} \langle \pi, \overline{F}_0(I) \rangle$

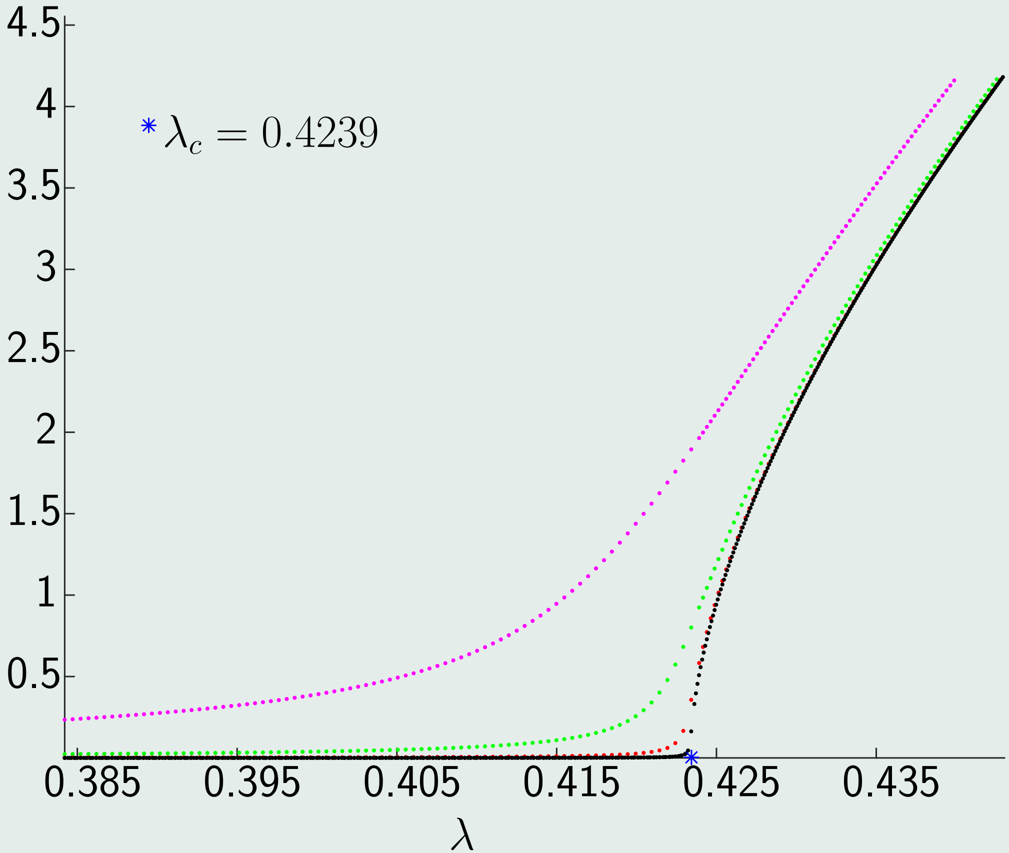
Let I_f be a fixed extremity.
We construct I_0 by integrating the uncontrolled flow on $[0, \tau_d]$.



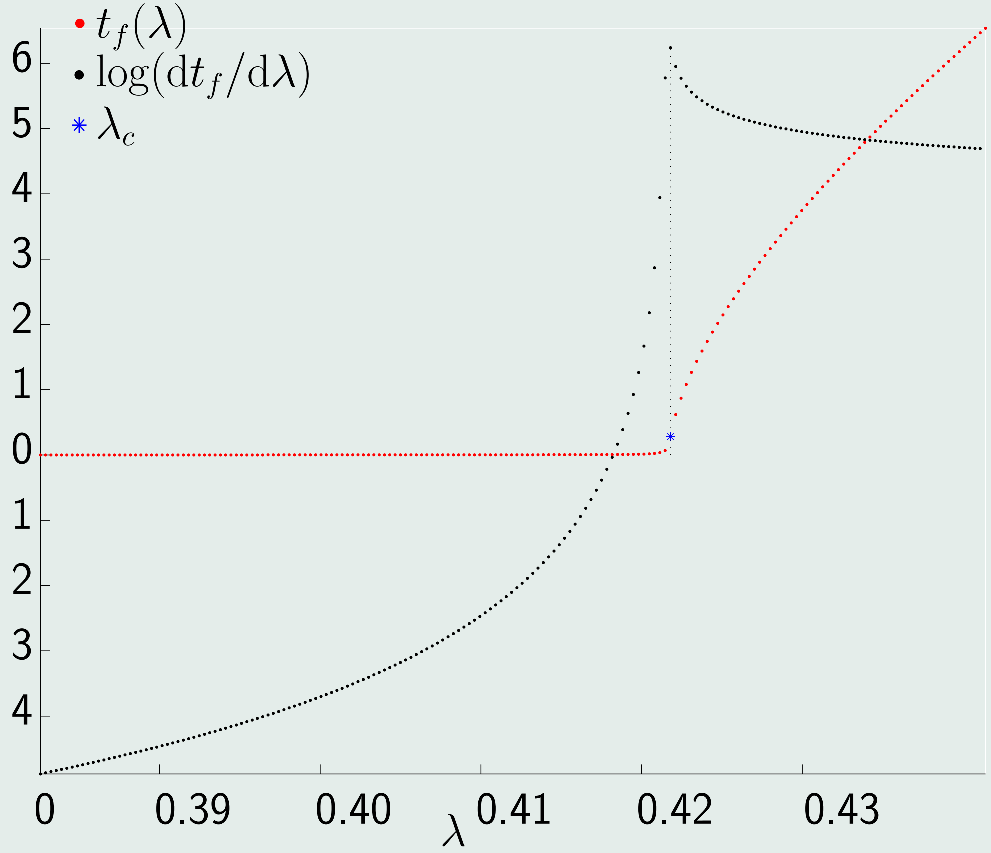
Boundary value problems

	Average BVP	Non average BVP
Extremal system	$\overline{K}^\lambda$	$H/\varepsilon = K^\lambda + p_\varphi \omega(I,\varphi)/\varepsilon$
Free parameters	$\lambda,\ \tau_d$	$\lambda,\ \tau_d,\ \varepsilon$
Boundary values	$I(0) = I_0, I(\tau_f) = I_f$	$I(0) = I_0, I(t_f) = I_f$
	$\overline{K}^\lambda = 1$	$p_\varphi(0) = 0, p_\varphi(t_f) = 0, H = \varepsilon$

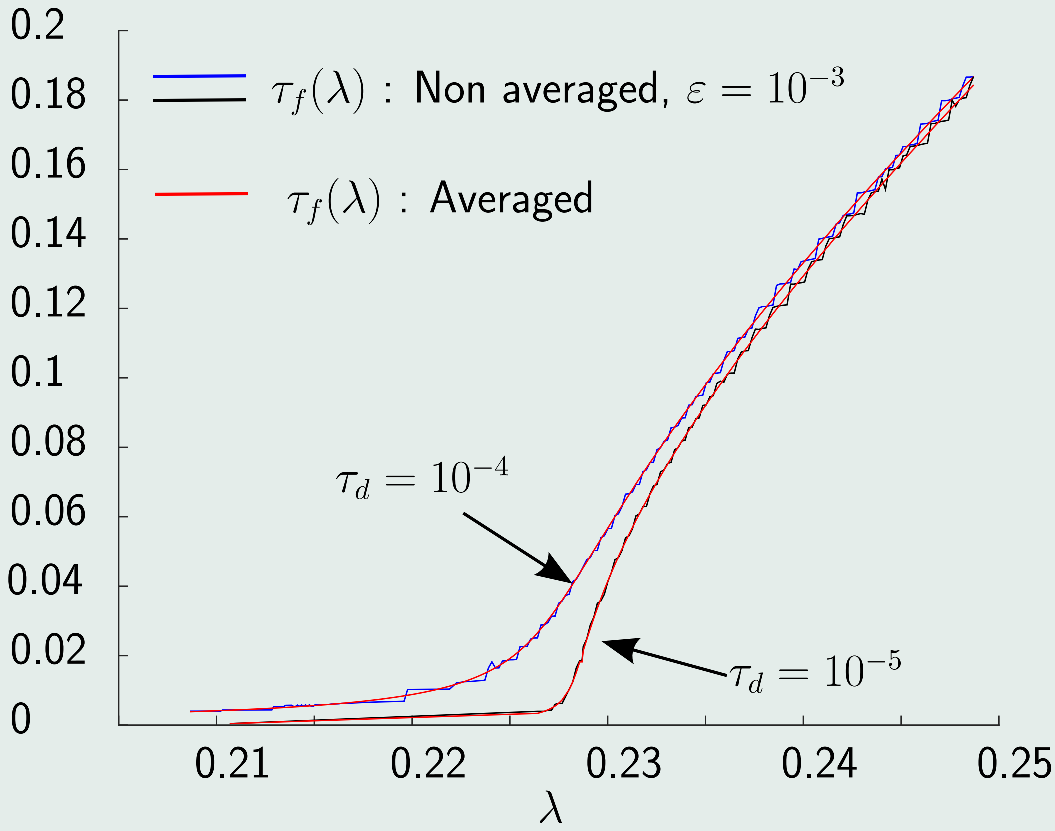
Settings We take ε small and analyze the dependance on λ when $\tau_d \to 0$ for both systems.



Curves $\lambda \mapsto \tau_f(\lambda)$ where $\tau_d \in \{$1e-2,1e-3,1e-4,1e-5$\}$ for the average system.



Minimum time w.r.t. to λ for the average system
and its derivative w.r.t. λ.



Curves $\lambda \to \tau_f$ for the average and non average system
where $\varepsilon = 10^{-3}$ et $\tau_d \in \{$1e-4,1e-5$\}$.

- relate the metric property to controllabilty property

- continuity of the value function at the critical value λ_c.

- averaging with several angles (and other perturbations): resonances