Sub-Riemannian geometry, Hamiltonian dynamics, micro-swimmers, Copepod nauplii and Copepod robot

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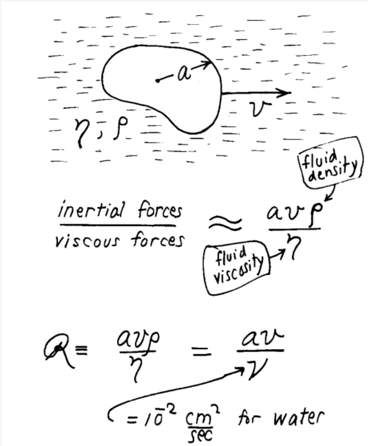
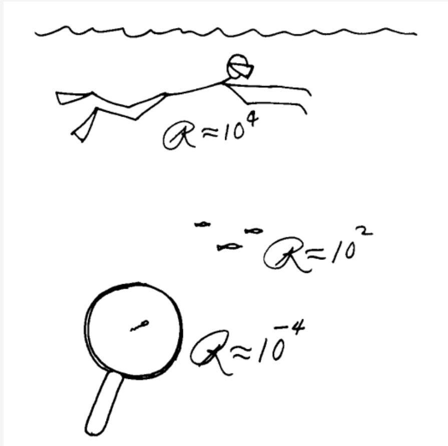
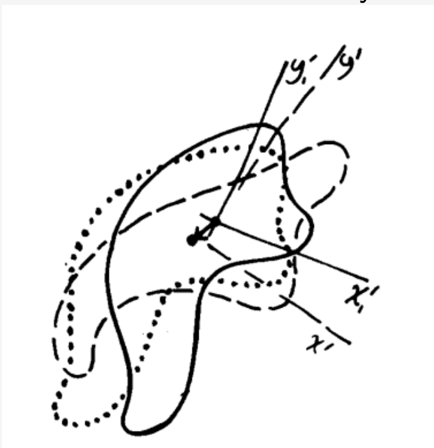
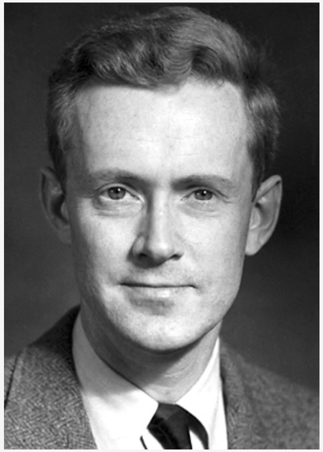




Reynolds number

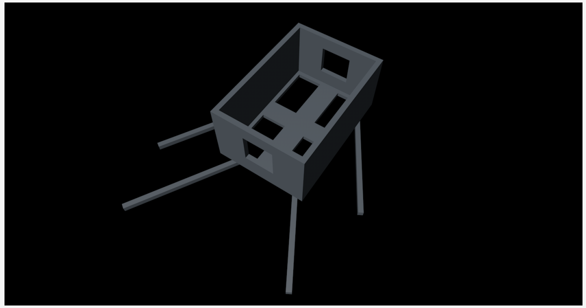




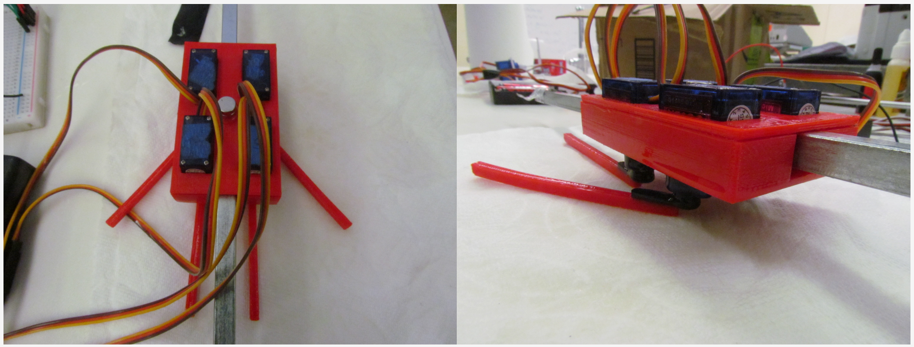
Shape deformations of a microswimmer

Copepod

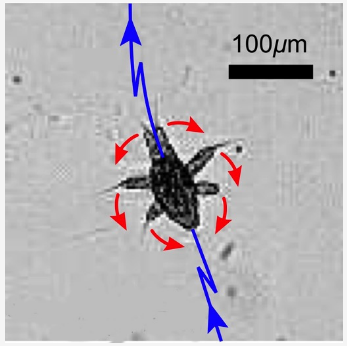
Copepod team in Hawaii



3D Printed copepod

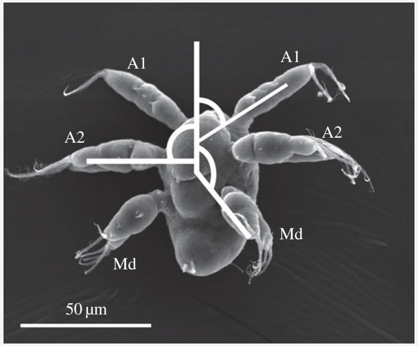


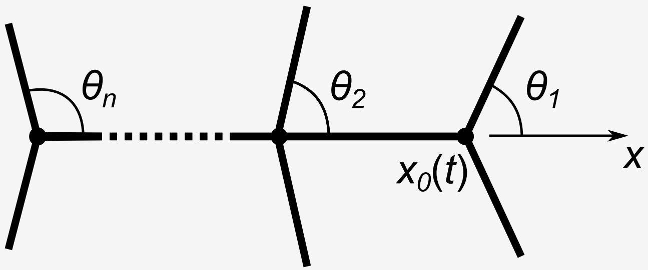
Copepod robot



Copepod observation

Validation of the model by observation



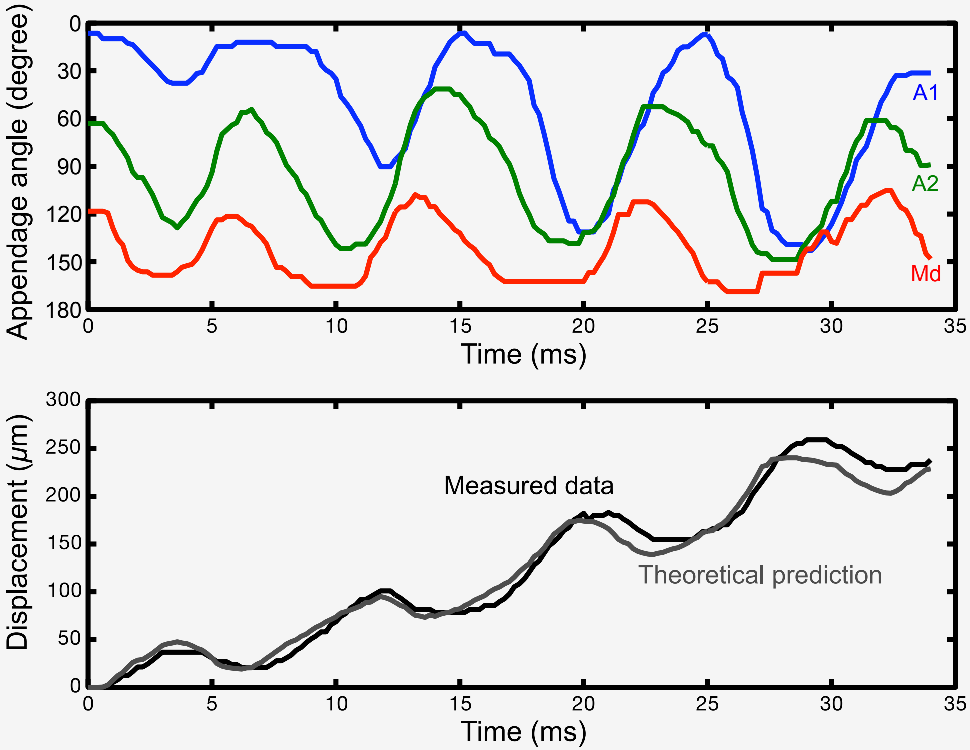


$$\dot{x}_0 = \frac{\sum_{i=1}^{n} \dot{\theta}_i \sin\theta_i}{\ell + \sum_{i=1}^{n}(1+\sin^2\theta_i)}$$

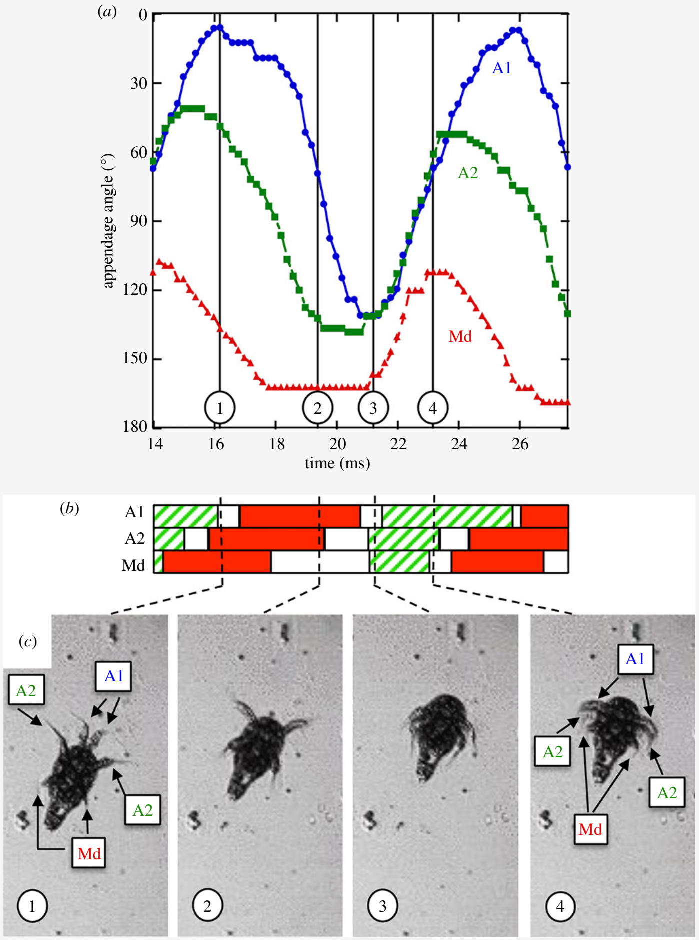
References

— D. Takagi, Swimming with stiff legs at low Reynolds number, *Phys. Rev. E 92.* (2015)

— P.H. Lenz, D. Takagi, D.K. Hartline, Choreographed swimming of copepod nauplii, *Journal of The Royal Society Interface* **12**, 112 20150776 (2015)

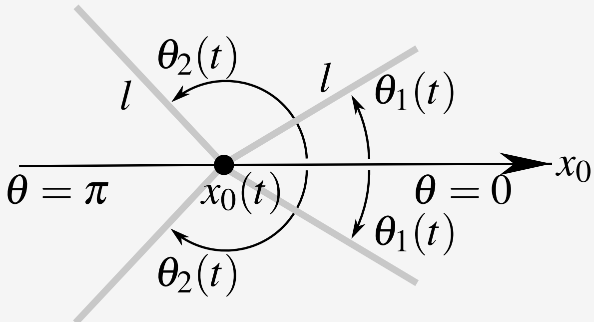


Model input (Top) and model prediction of naupliar displacement (Bottom). (Top) Lines show the angular position of three appendages at 0.2 ms intervals starting from rest (T=0 ms) to completion of fourth return stroke (T=32 ms) from an observed swim episode. Top line : A1 (blue) ; middle line : A2 (green) ; and bottom line : Md (red). (Bottom) Copepod displacement over time : observed (black line) and theoretical model prediction (grey).



Measured movements of a larval copepod. Panel (a) shows variations over time of the orientation angles of three leg pairs, labeled as A1, A2, Md. Panel (b) shows time intervals when each leg pair performs a power stroke (red), a return stroke (green stripes), or remains stationary (white). Panel (c) shows snapshots of the copepod at four representative times.

2-link symmetric swimmer.



Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i,\ i = 1,2 \quad (\text{state constraint} : 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Introduce a cost

Mechanical energy $=$ Work of the Drag forces
Compare different strokes and different swimmers.

Criterion. minimize drag forces : $\dot{q}S(q)\dot{q}^{\mathsf{T}}$, $\boldsymbol{q} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{x}_0)$ and S is positive definite $\Longrightarrow$ quadratic form in (u_1, u_2).

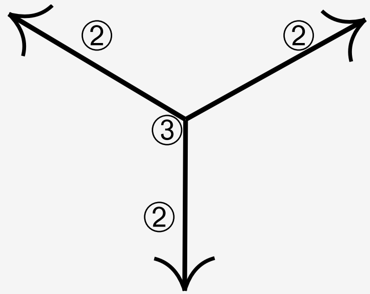
— Dido's problem : a rough model of a swimmer.
Normal extremals :

$$x(t) = \frac{A}{\lambda}(\sin(\lambda t + \phi) - \sin\phi)$$
$$y(t) = \frac{A}{\lambda}(\cos(\lambda t + \phi) - \cos\phi)$$
$$z(t) = \frac{A^2}{\lambda}t - \frac{A^2}{\lambda^2}\sin(\lambda t)$$

with $A = \sqrt{H_1^2 + H_2^2}$ and ϕ is the angle of the vector $(\dot{x}, -\dot{y})$ at the origin.

— Generalization by Agrachev/Montgomery :

Charged particle in a magnetic field $\Leftrightarrow$ 2D Riemannian metric



Example of cut locus on S^2.

Simple branch : two intersecting minimizers,
Ramification point : three intersecting minimizers.

References

— A. Agrachev, J.P. Gauthier On the Dido problem and plane isoperimetric problems, *Acta Appl. Math.* **57**, 3 (1999) 287–338

— R. Montgomery, Isoholonomic problems and some applications, *Commun. Math. Phys.* **128**, 3 (1990) 565–592

The swimming curvature (Alouges, Avron-Raz)

Associated displacement produced by this stroke γ :

$$x_0(T) - x_0(0) = \oint_\gamma \omega$$

where ω is the smooth one form

$$\omega = \sum_{i=1}^{2} \frac{\sin\theta_i}{\Delta(\theta)} \mathrm{d}\theta_i.$$

$$x_0(T) - x_0(0) = \oint_\gamma \omega = \int_D \mathrm{d}\omega.$$

References

— F. Alouges, A. DeSimone, A. Lefebvre, Optimal strokes for low Reynolds number swimmers : an example, *J. Nonlinear Sci.* **18**, 277–302 (2008)

— J.E. Avron and O. Raz, A geometric theory of swimming : Purcell's swimmer and its symmetrized cousin, *New Journal of Physics* **10**, 6 (2008) : 063016

Lemma 1.

1. $\mathrm{d}\omega = -f(\theta)\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$ $f(\theta) = (2\sin\theta_1 \sin\theta_2(\cos\theta_1 - \cos\theta_2))/\Delta(\theta)^2$
2. $\mathrm{d}\omega < 0$ *in the interior of the triangle* $\mathscr{T}$, *and* $\mathrm{d}\omega$ *vanishes on the boundary of* $\mathscr{T}$.

Geometric consequence. Restricted to the interior of $\mathscr{T}$, $\mathrm{d}\omega$ is a volume form (density) which allows to estimate the displacement associated to *small (amplitudes) strokes*. It can be "normalized" using the 2-form associated to a general Riemannian metric :

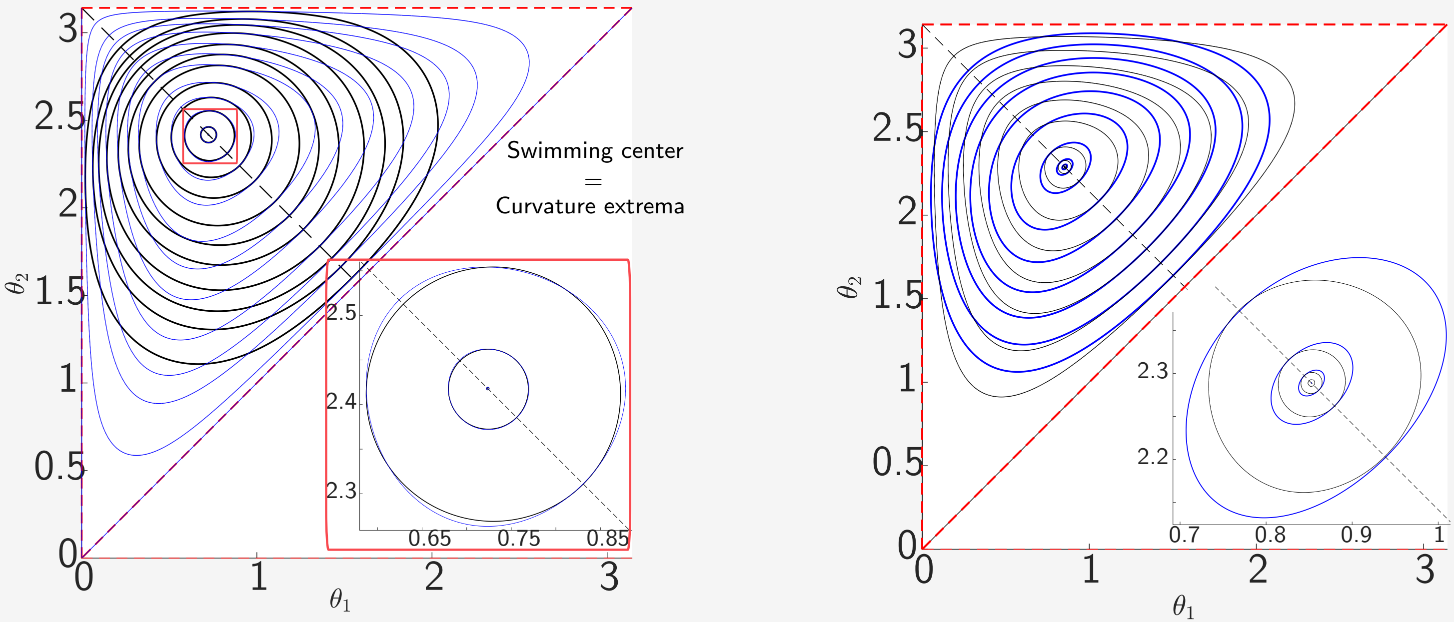
$$g : E(\theta)\,u^2 + 2G(\theta)\,uv + F(\theta)\,v^2$$

defined as

$$\omega_g = \sqrt{EF - G^2}\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2. \qquad (*)$$

1. A geometric 2-link micro-swimmer is defined by $(\mathrm{d}\omega, g)$.
2. The swimming curvature is defined as the ratio :

$$SK = \frac{\mathrm{d}\omega}{\omega_g} = -\frac{f(\theta)}{\sqrt{E(\theta)F(\theta) - G(\theta)^2}}.$$

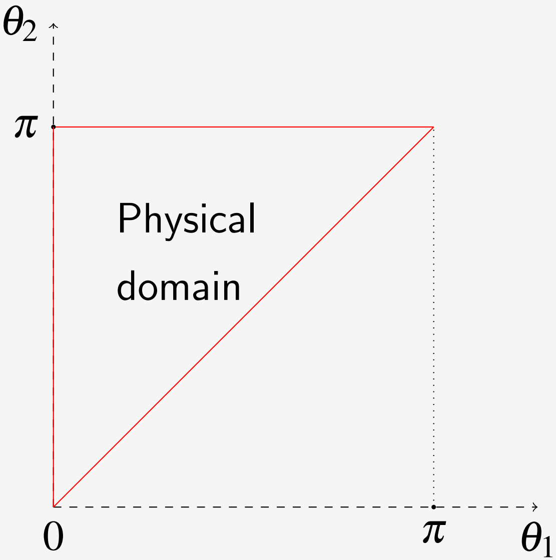


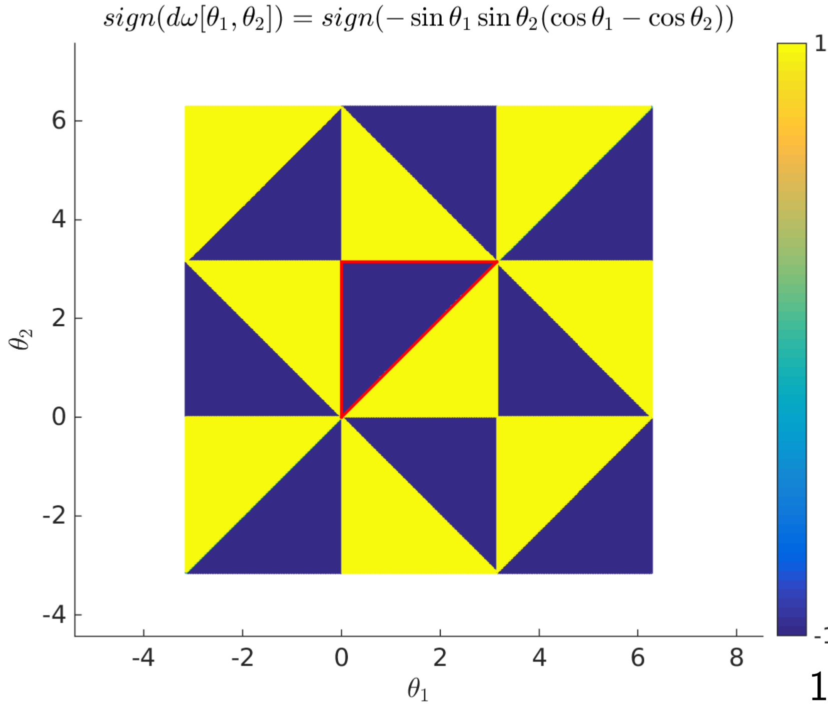
Level-sets of the swimming curvature *(blue)* and family of simple strokes *(black)* for the Euclidean metric (top) and the mechanical cost (bottom).

- Compute **optimal strokes** minimizing an efficiency.

- Stroke : T-**periodic motion of the shape variables** θ s.t. $x_0(T) - x_0(0) > 0$.

- Admissible stroke : find closed curves in the θ-plane **contained in the triangle** :





$$\mathscr{E}(q(\cdot)) = \textbf{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR}.

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1u_2 + c(q)\,u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t$$

Two steps.

$$\max_{u(\cdot),\,x_0(T)} \mathscr{E} \Leftrightarrow \left(\begin{array}{c} \dot{q} = \sum_{i=1}^{2} u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t, \\ \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t \ \text{ with } \boldsymbol{x_0(T)} \textbf{ fixed}\text{, then } \underbrace{\text{select} \max_{x_0(T)} \mathscr{E}}_{\text{Transversality cond.}} \end{array} \right)$$

Compute points on the Sub-Riemannian sphere $\rightarrow$
provide candidate solutions for the maximum of efficiency problem.

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T),\ i = 1,2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2$$

Boundary value problem

$$\begin{cases} \dot{z} = \overrightarrow{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \ \textbf{(fixed)} \\ \theta_i(0) = \theta_i(T),\ i = 1,2 \\ p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2 \end{cases},$$

— `HamPath`, *indirect method :* based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .

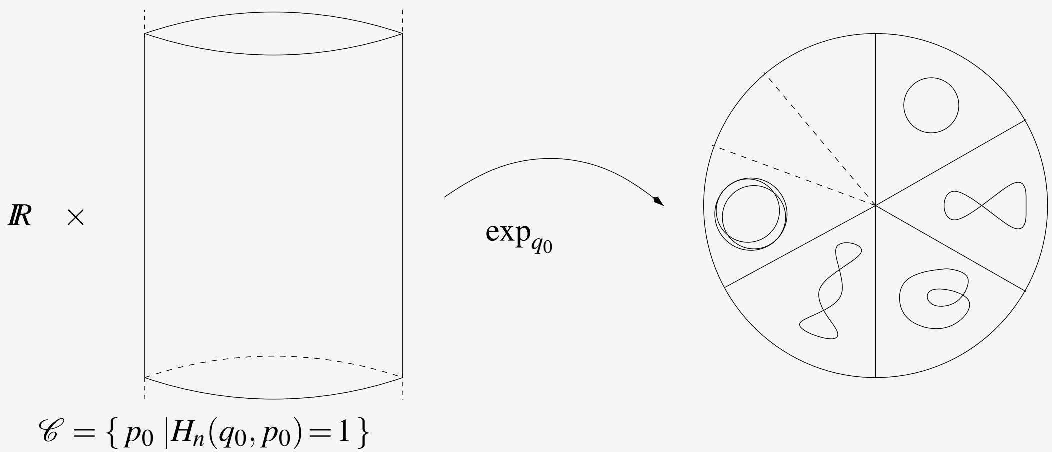
— `Bocop`, *direct method :* gives an initialization for the shooting algorithm of `HamPath`.

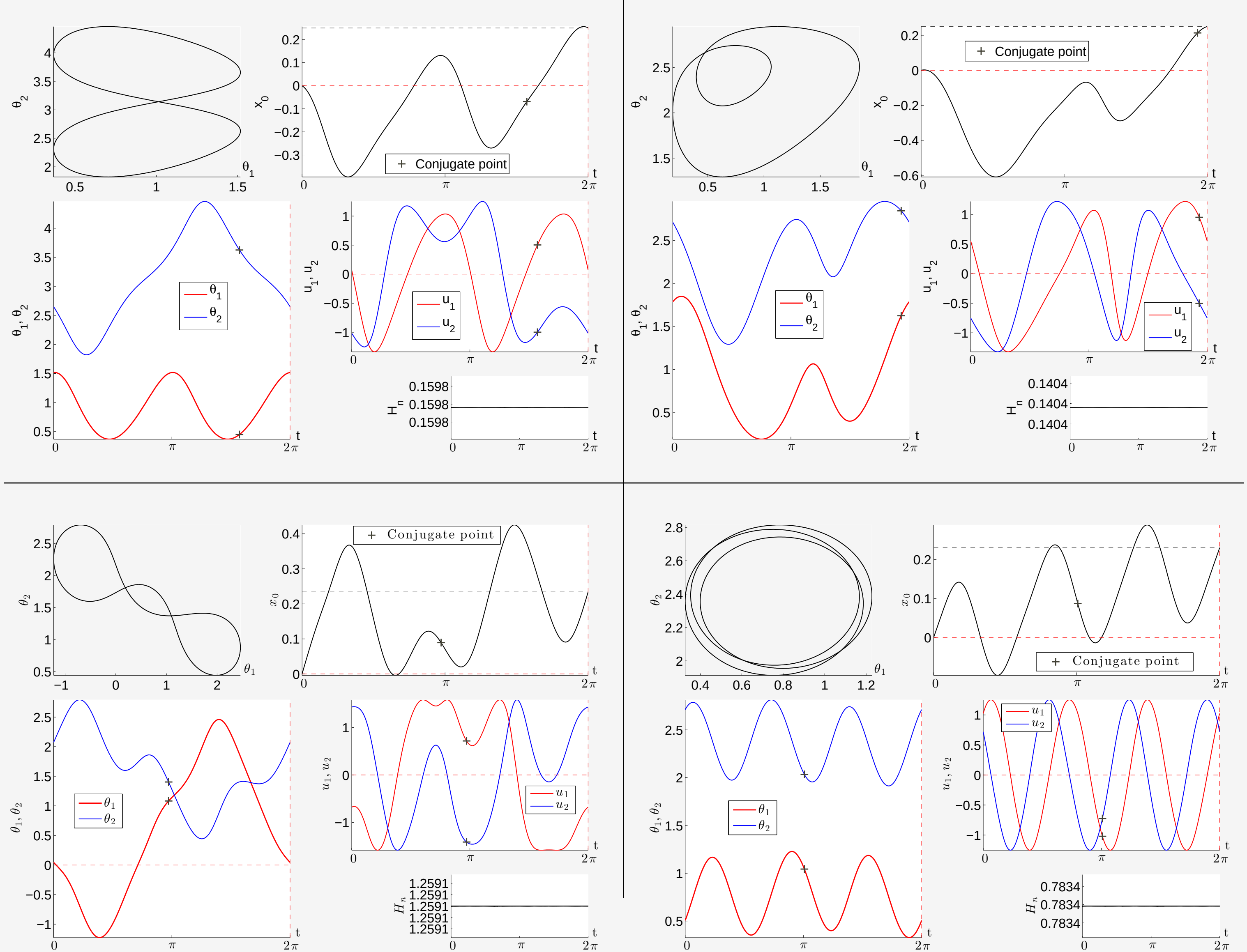
First conjugate time t_c : The first time t_c when the exponential mapping

$$\exp_{q_0} : \mathbb{R} \times \mathscr{C} \to M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

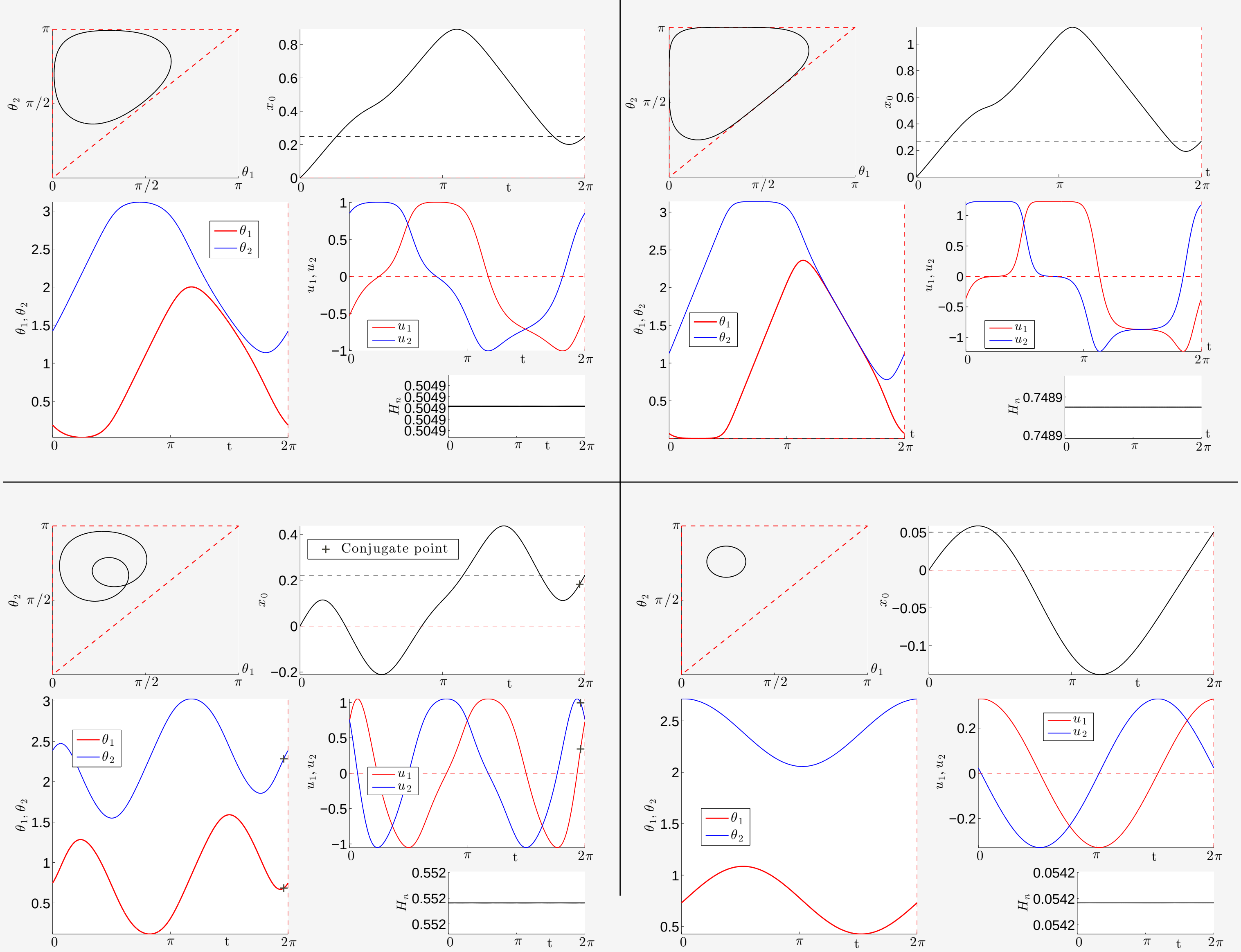
is not an immersion at (t_c, p_0).

Theorem 2. *Let $q : [0, T] \longrightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(.)$ has at least one conjugate point on $]0, T[$, then $q(\cdot)$* **is not a local minimizer** *for the L^∞ topology on the controls considering the optimal control problem with* **fixed extremities***.*



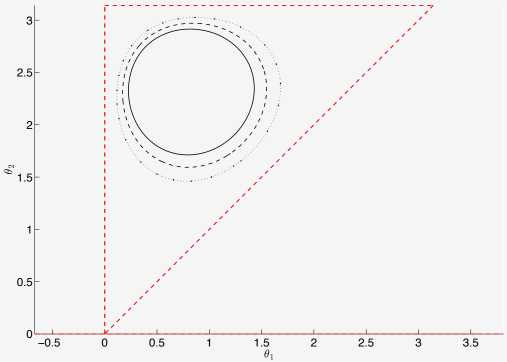


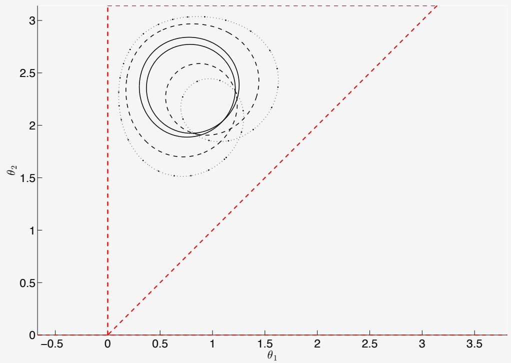
+ Conjugate point
θ_1
θ_2
u_1
u_2
x_0
H_n
0.1598
0.1404
1.2591
0.7834

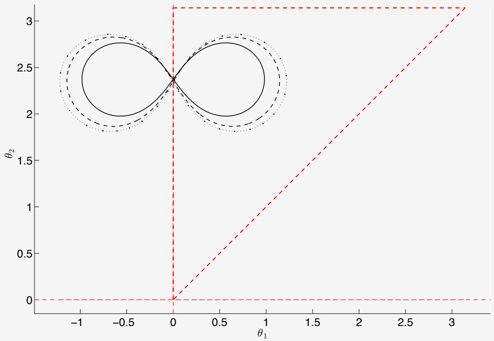


θ1
θ2
π/2
π
t
x0
θ1, θ2
u1, u2
Hn
0.5049
0.7489
+ Conjugate point
0.552
0.0542

Generate normal strokes







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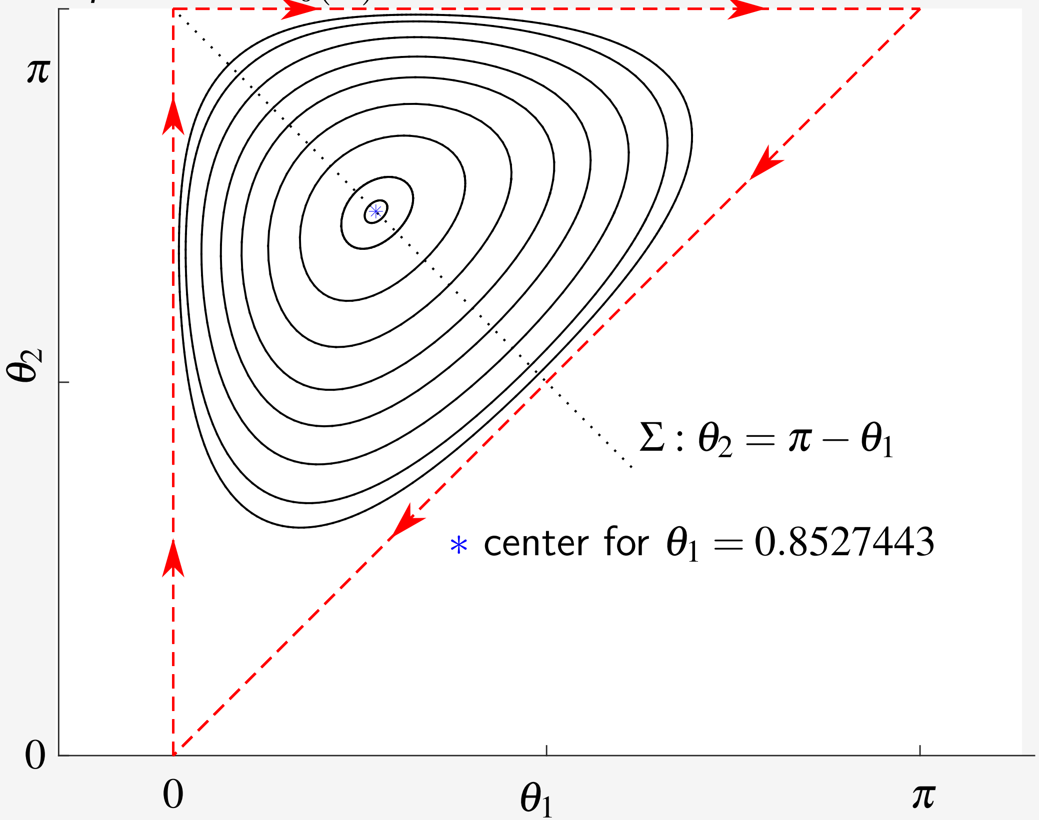
$$\int_0^T L(q,u) \leftarrow \min_{u(\cdot)}$$

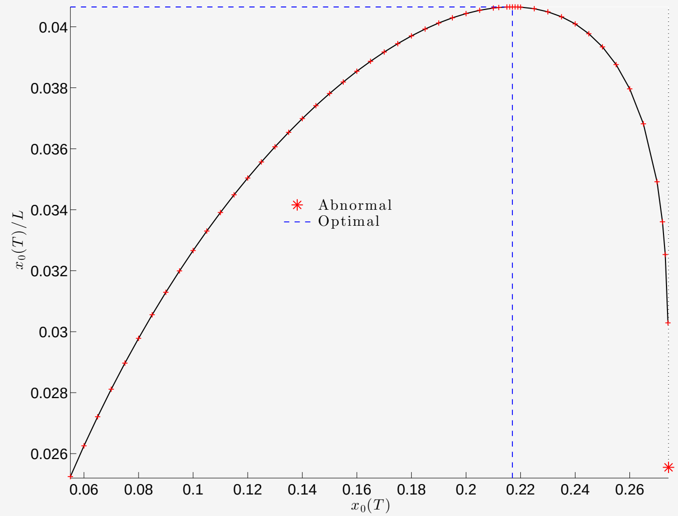
Proposition 3. *There is a one parameter family of normal simple loop strokes parameterized by the displacement $x_0(T)$.*



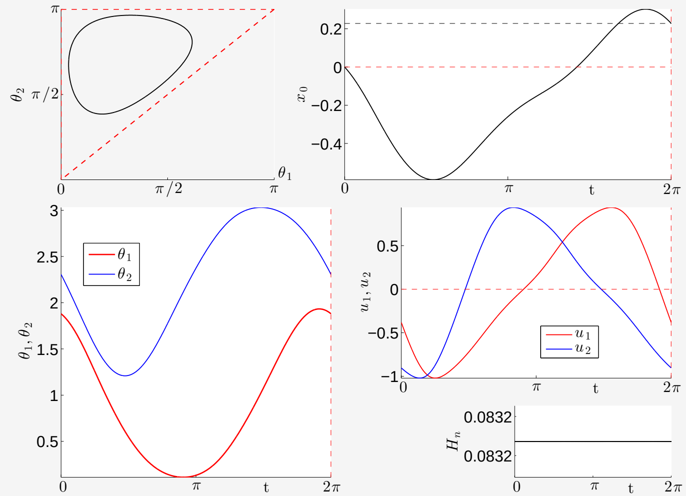
Geometric efficiency : $\mathscr{E} = x_0(T)/l_{SR}(q)$, $\quad$ **($\boldsymbol{x_0(T)}$ free)**

Transversality condition of the maximum Principle

$$p_{x_0}(T) = q^0(T)/x_0(T), \qquad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

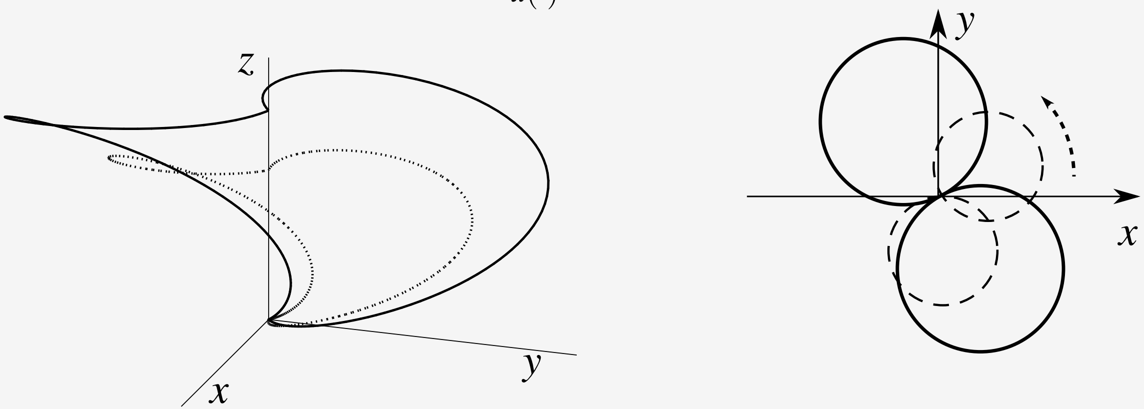
Theorem 4. *The abnormal stroke is not minimizing.*

Normal form of order -1 Nilpotent model of order -1, the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

$$u_1^2 + u_2^2 \leftarrow \min_{u(\cdot)}, \quad \dot{q} = u_1\hat{F}(q) + u_2\hat{G}(q)$$



Geodesics $q(\cdot)$ for the model of order -1.

These families of circles are **not generic**

Theorem 5 (Brockett, Chakir et al). *The model of order -1 is equivalent to the model of order 0.*

Application to the Copepod z cannot be identified to the displacement x_0.
$x = \theta_1 - \theta_{10},\ y = \theta_2 - \theta_{20},\ \theta_{20} = \pi - \theta_{10}$. (contact point)

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x,y,z) = \frac{\partial}{\partial x} + \left(a_{11}(\theta_{10})\,xy + a_{02}(\theta_{10})\,y^2 + y\right)\frac{\partial}{\partial z}$$
$$F_2(x,y,z) = \frac{\partial}{\partial y} + \left(a_{11}(\theta_{10})\,xy + a_{02}(\theta_{10})\,x^2 - x\right)\frac{\partial}{\partial z}$$

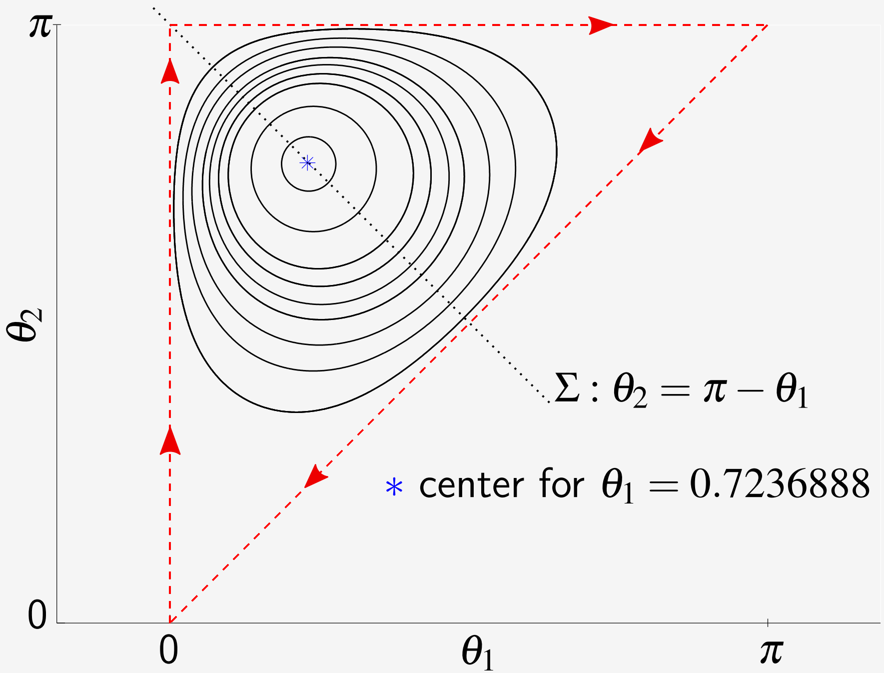
$$(F_1, F_2) \quad \underset{\varphi}{\Longleftrightarrow} \quad (\hat{F}, \hat{G})$$

Lemma 6. *The only value of θ_{10} such that the transformation φ doesn't mix up the* **shape variables (x,y) and the displacement z** *corresponds to the center of swimming of the family.*

$$\varphi(x,y,z) = (x + c_{011}(\theta_{10})\,\boldsymbol{yz} + c_{001}(\theta_{10})\,\boldsymbol{z},\ y - c_{011}(\theta_{10})\,\boldsymbol{xz} + c_{001}(\theta_{10})\,\boldsymbol{z},\ z + P(x,y,z,\theta_{10}))$$

$$\underset{\text{Lem. }5}{\Longrightarrow} c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0 \quad \Leftrightarrow \quad \theta_{10} = \text{center of the family.}$$

Theorem 7. *The center of swimming is a SR-invariant.*



One parameter family of simple loops with different metrics.

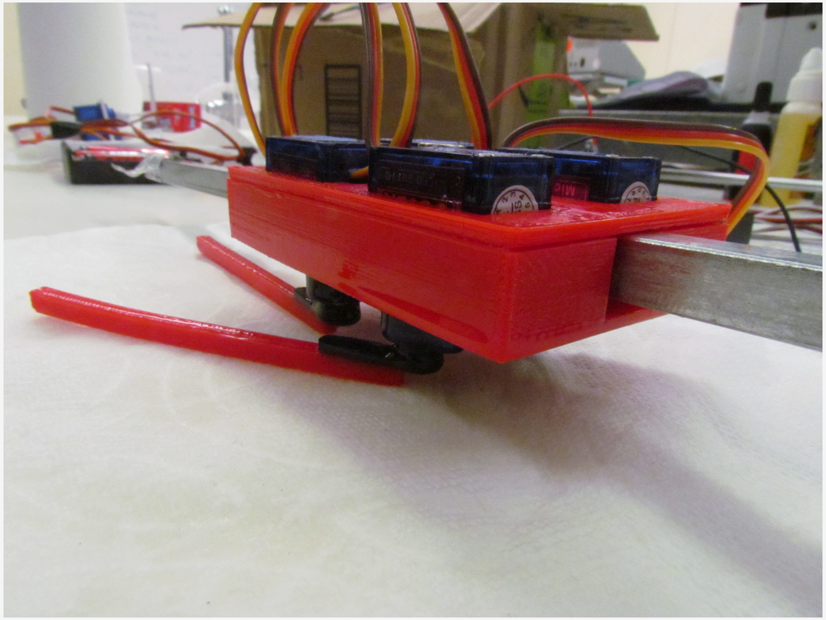
The family can be obtained by numerical continuation from the center of swimming.

Theoretical aspect Use the model of order 1 to compute the conjugate locus by continuation methods.

$$F_1(x,y,z) = \frac{\partial}{\partial x} + \frac{y}{2}(1+Q(x,y))\frac{\partial}{\partial z},$$
$$F_2(x,y,z) = \frac{\partial}{\partial y} - \frac{x}{2}(1+Q(x,y))\frac{\partial}{\partial z}$$

where

$$Q(x,y) = -0.7165898586\,x^2 - 0.7379854942\,y^2.$$



Validation of the adequation between the predicted and observed most efficient stroke

Second order sufficient optimality conditions

for a Copepod stroke

$$\begin{cases} \min & J(q(.),u(.)) := c(q(0),q(T)) \\ \text{s.t.} & \dot{q}(t) \;=\; F(q(t),u(t)) \\ & m(q(0),q(T)) = 0\,, \end{cases}$$

$H(q,u,p) := p \cdot F(q,u)$ and $h(q_0,q_T) := c(q_0,q_T) + \nu \cdot m(q_0,q_T)$.
Consider a **normal extremal** $(\overline{q},\overline{p})$ associated with $\overline{u}$.

Second variation

$$\delta^2 J(\delta q(.),\delta u(.)) := 1/2\,[\delta q(0)^\intercal \quad \delta q(T)^\intercal]\,\boldsymbol{C}\,[\delta q(0) \quad \delta q(T)]^\intercal$$
$$+\,1/2\int_0^T (\delta q(t)^\intercal \boldsymbol{\partial}_{qq}\boldsymbol{H}(t)\delta q(t) + 2\,\delta q(t)^\intercal \boldsymbol{\partial}_{qu}\boldsymbol{H}(t)\delta u(t) + \delta u(t)^\intercal \boldsymbol{\partial}_{uu}\boldsymbol{H}(t)\delta u(t))\,\mathrm{d}t$$

where $\quad \boldsymbol{C} := \mathrm{Hessian}_{q_0,q_T}(h)$.

$$\begin{cases} \min & \delta^2 J(\delta q(.), \delta u(.)) \\ \text{s.t.} & \dot{\delta q} := \boldsymbol{\partial}_q \boldsymbol{F}(t)\, \delta q(t) + \boldsymbol{\partial}_u \boldsymbol{F}(t)\, \delta u(t) \\ & \nabla_{q_0} m(\overline{q}(0), \overline{q}(T))\, \delta q(0) + \nabla_{q_T} m(\overline{q}(0), \overline{q}(T))\, \delta q(T) = 0 \end{cases}$$

Classical optimality conditions

- *2nd order necessary conditions :* $\delta^2 J(\delta q(.), \delta u(.)) \geq 0$,
- *2nd order sufficient conditions :* $\delta^2 J(\delta q(.), \delta u(.))$ coercive.

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s) = \boldsymbol{Z}\,\Phi(t,s) \\ \Phi(s,s) = \mathrm{Id}, \end{cases}$$

where

$$Z := \begin{bmatrix} \partial_q F - \partial_u F [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_u F [\partial_{uu} H]^{-1} \partial_u F^T \\ -\partial_{qq} H + \partial_{qu} H [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_q F [\partial_{uu} H]^{-1} \partial_u F^T \end{bmatrix}.$$

Define

$$\mathscr{W} := \begin{bmatrix} \phi_{22}\phi_{12}^{-1} & \phi_{21} - \phi_{22}\phi_{12}^{-1}\phi_{11} \\ -\phi_{12} & \phi_{12}^{-1}\phi_{11} \end{bmatrix}, \quad \Phi(0,T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 8 (**Standard conditions**). *Assume*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H}(t) \leq -\varepsilon$ *Id on* $[0,T]$, $\overline{u}(.)$ *bounded and* $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.), \boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), \overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *t.q.*

$$\left[\xi_0^T \ \xi_1^T\right] \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2 ,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0.$$

Then $(\overline{q}(.), \overline{u}(.))$ *is a* $W^{1,\infty}$*-minimizer and* **locally unique**.

Boundary values

$$\theta_j(0) = \theta_j(T) \quad j = 1,2,$$
$$x_0(0) = 0,\ x(T) = x_T, \quad x_T \text{ is fixed}$$

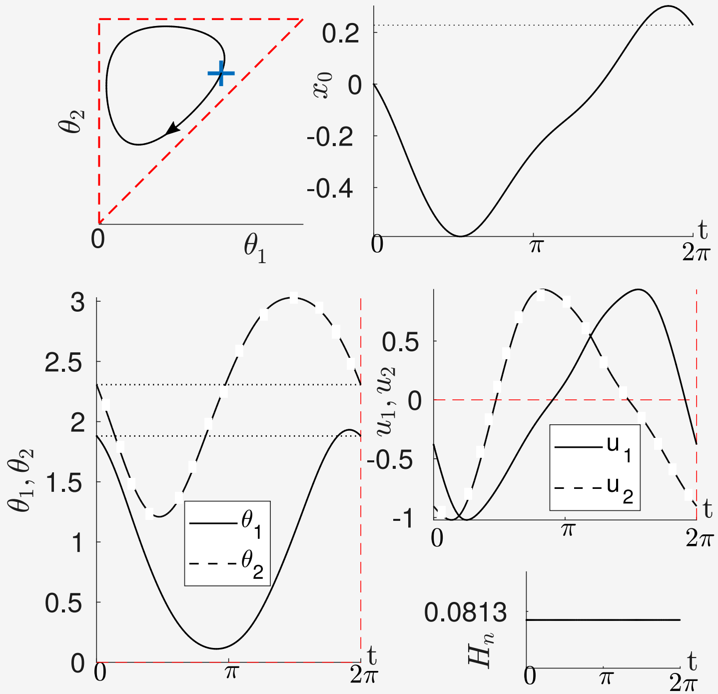
Proposition 9. *Take* $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ *and let* $(\overline{q}(\cdot), \overline{u}(\cdot), \overline{p}(\cdot))$ *be a normal extremal. For all* $a \in I$ *and* $t \in [0,T]$, *we define* $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot))$, $u_1^a(\cdot), u_2^a(\cdot)$ *and* $p^a(\cdot)$ *by*

$$\theta_j^a(t) = \overline{\theta}_j(t+a), \quad u_j^a(t) = \overline{u}_j(t+a) \quad \text{for } j = 1,2,$$
$$x^a(t) = \overline{x}(t+a) - \overline{x}(a),\ p^a(t) = (\overline{p}_1(t), \overline{p}_2(t+a), \overline{p}_3(t+a)).$$

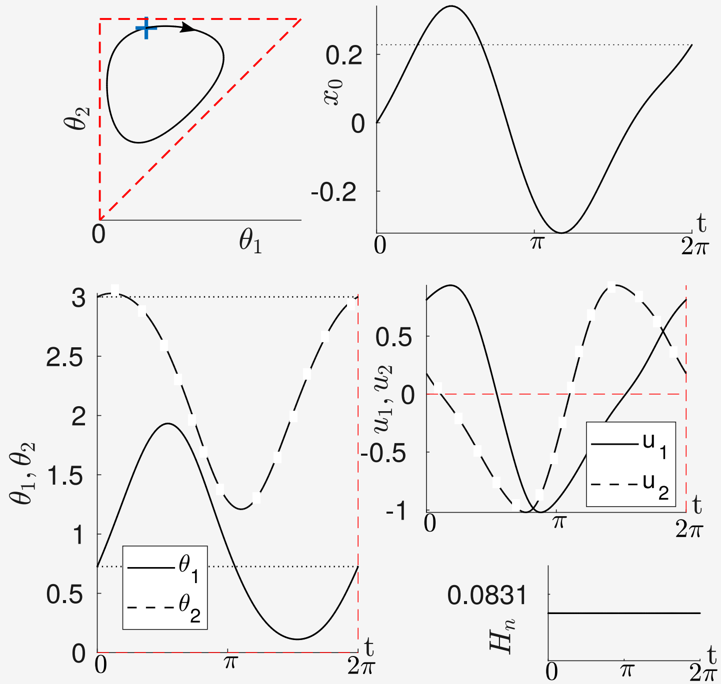
Then, for $\varepsilon > 0$ *small enough, the normal extremal* $(\overline{q}(.), \overline{p}(.), \overline{u}(.))$ *is continuously embedded in the family of extremals* $(q^a(.), p^a(.), u^a(.))_{a \in I}$.

These strokes have the SAME COST and satisfy the SAME BOUNDARY CONDITIONS

$\Longrightarrow$ **Standard conditions fail** because of non-unique minimizers.



θ_2
0
θ_1
0.2
0
-0.2
-0.4
x_0
0
π
2π
t
3
2.5
2
1.5
1
0.5
0
θ_1, θ_2
θ_1
θ_2
0
π
2π
t
0.5
0
-0.5
-1
u_1, u_2
u_1
u_2
0.0813
H_n



θ_2
0
θ_1
0.2
0
-0.2
x_0
0
π
2π
t
3
2.5
2
1.5
1
0.5
0
θ_1, θ_2
θ_1
θ_2
0.5
0
-0.5
-1
u_1, u_2
u_1
u_2
0.0831
H_n

Theorem 10 (**Refined Conditions**, Gavriel, Vinter (2014)). *Assume the reference normal extremal* $(\overline{q}(.), \overline{u}(.), \overline{p}(.))$ *is continuously embedded in a family of extremals and*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H} \leq -\varepsilon$ *Id on* $[0,T]$, $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.), \boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), \overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *s.t.*

$$[\xi_0^T \ \xi_1^T] \, \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2 ,$$

for all vectors $\xi_0, \xi_1 \neq 0$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0 \qquad \textit{and} \qquad \boldsymbol{\Gamma}^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0 \, .$$

where $\Gamma := \left[\begin{array}{c} \nabla_a q^a(0) \\ \nabla_a q^a(T) \end{array} \right] \Big|_{a=0}$.

Then $(\overline{q}(.), \overline{u}(.))$ *is a* **local** $W^{1,\infty}$**-minimizer**.

Computation. Define the matrix N_s from the subspace $\mathscr{L}_s$ s.t.

$$\mathscr{L}_s = \{\, (\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \nabla_{q_0,q_T} m(q_0, q_T)\, (\xi_0 \quad \xi_T)^{\mathsf{T}} = 0\,\} =: \mathrm{Im}(N_s)$$

Standard conditions. Does the matrix $\mathscr{W}_s := N_s^{\mathsf{T}}\,(\mathscr{W}^{\mathsf{T}} + \mathscr{W})\,N_s \in \mathscr{M}_2$ is positive-definite ?

Consider

$$\Gamma_r = \big(\nabla_a q^a(0) \;\; \nabla_a q^a(T)\big)_{a=0} = \big(\dot{q}(0) \;\; \dot{q}(T)\big)$$

and the linear subspace $\mathscr{L}_r$ s.t.

$$\mathscr{L}_r := \mathscr{L}_s \cap \{(\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \Gamma_r^{\mathsf{T}}\,(\xi_0 \quad \xi_T)^{\mathsf{T}} = 0\} =: \mathrm{Im}(N_r)$$

Refined conditions. Does $\mathscr{W}_r := N_r^{\mathsf{T}}\,(\mathscr{W}^{\mathsf{T}} + \mathscr{W})\,N_r > 0$?

Relative tolerance	(Standard condition) $\mathrm{Spec}(W_s)$	(Refined condition) $\mathrm{Spec}(W_r)$
10^{-5}	**6.89e-4** 3.42	22.5
10^{-8}	**-9.12e-7** 3.42	22.5

— Standard conditions fail : $\mathscr{W}_s$ has a zero eigenvalue.
— **BUT** Refined conditions are satisfied : $\mathscr{W}_r$ is positive-definite.

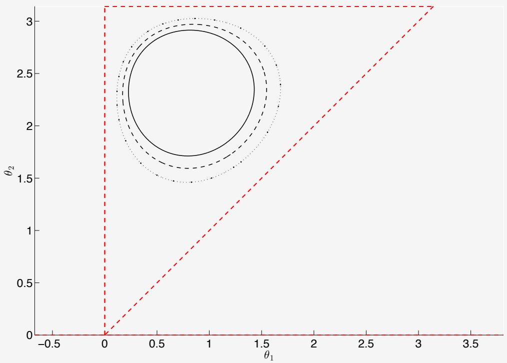
Theorem 11 (Numerical)**.** *The simple loop normal stroke* $(\overline{q},\overline{u})$ *is* $W^{1,\infty}-optimal$.

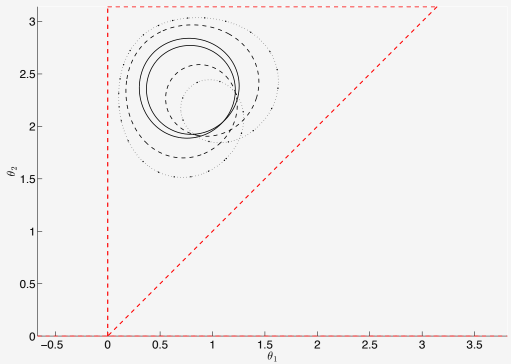
— *Contact point* : expression of the generic normal form at any point inside the triangle $\rightarrow$ unique family of simple loops.

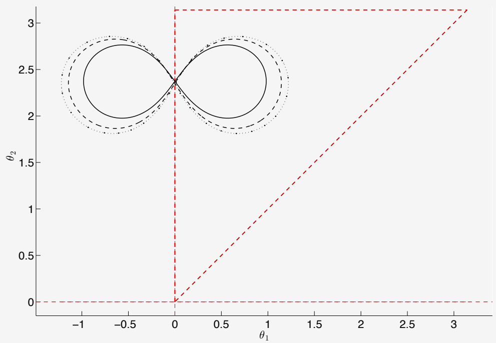
— *Martinet point* : compute the normal form for a point on the edges $\rightarrow$ locate the eight loops.

— estimation of the first conjugate time using normal forms.

— swimmer model with more than 2 pairs of links.


θ_2
θ_1

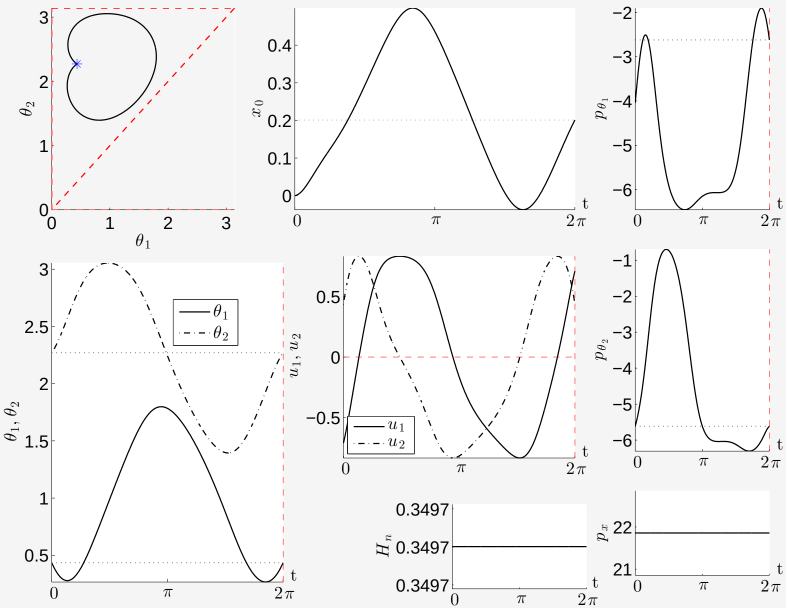

θ_2
θ_1


θ_2
θ_1

$$\mathscr{E}' = \frac{x_0(T)\,\boldsymbol{m(\theta(0))}}{l(q)}, \quad (m \text{ smooth})$$

Transversality condition of the maximum Principle

$$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial \mathscr{E}'}{\partial \theta(0)}$$



Theorem 12 (Chakir, Gauthier, Kupka, 1996)**.** *The* **generic model** *is given by the normal form of order* 1

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z},\; G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q *quadratic in* (x,y)*.*

Remark 13. *This normal form can be used to approximate the one parameter family of simple strokes.*